UNIT-II Numerical Differentiation, Numerical integration: Trapezoidal rule and Simpson's 1/3rd and 3/8 rules. Solution of Simultaneous Linear Algebraic Equations by Gauss's Elimination, Gauss's Jordan, Crout's methods, Jacobi's, Gauss-Seidal, and Relaxation method.,

Numerical Differentiation:

Differentiate Newton's forward interpolation formula with respect to "p" we get following

Newton's forward difference formula:

$$f'(x)=f'(a+ph)=\frac{1}{h}\left[\Delta f(a)+\frac{2p-1}{2!}\Delta^2 f(a)+\frac{3p^2-6p+2}{3!}\Delta^3 f(a)+\frac{4p^3-18p^2+22p-6}{4!}\Delta^4 f(a)+..........\right]$$

$$f''(x)=f''(a+ph)=\frac{1}{h^2}\left[\Delta^2 f(a)+(p-1)\Delta^3 f(a)+\frac{6p^2-18p+11}{12}\Delta^4 f(a)+..........\right]$$

When $x=x_0$ then $p=x-x_0/h=0$ hence these formulae reduce to

$$f'(x)=f'(a)=\frac{1}{h}\left[\Delta f(a)-\frac{1}{2}\Delta^2 f(a)+\frac{1}{3}\Delta^3 f(a)-\frac{1}{4}\Delta^4 f(a)+..........\right]$$

Newton's Backward difference formula:

$$f'(x)=f'(a+ph)=\frac{1}{h}\left[\nabla f(x_n)+\frac{2p+1}{2!}\nabla^2 f(x_n)+\frac{3p^2+6p+2}{6}\nabla^3 f(x_n)+\frac{4p^3-18p^2+22p-6}{4!}\nabla^4 f(x_n)+..........\right]$$

$$f''(x)=f''(a+ph)=\frac{1}{h^2}\left[+\nabla^2 f(x_n)+(p+1)\nabla^3 f(x_n)+\frac{6p^2+18p+11}{12}\nabla^4 f(x_n)+..........\right]$$

When $x=x_0$ then $p=x-x_0/h=0$ hence these formulae reduce to

$$f'(x)=f'(a)=\frac{1}{h}\left[\nabla f(x_n)+\frac{1}{2}\nabla^2 f(x_n)+\frac{1}{3}\nabla^3 f(x_n)-\frac{1}{4}\nabla^4 f(x_n)+..........\right]$$

Numerical Differentiation

Q.1 Find $f'(x)$ at x=0.1 given that

x:	0.1	0.2	0.3	0.4
$f(x)$:	0.9975	0.9900	0.9776	0.9604

[RGPV. MAY 19]

Q.2 Find the first and second derivatives of f(x)at (i) x = 1.1 (ii) 1.6

x:	1	1.1	1.2	1.3	1.4	1.5	1.6
$f(x)$:	7.989	8.403	8.781	9.129	9.451	9.750	10.031

(i) 3.952, -3.7417 (ii)2.7477, -0.71 , [Dec. 2007, 11,June 2013]

Q.3 Find the first and second derivatives of f(x)at x = 1.1

x:	1	1.2	1.4	1.6	1.8	2.0
$f(x)$:	0	0.128	0.544	1.296	1.432	4

Ans: -0.5030, -24.13 **[June 2009, Feb. 2010, June 2014, Dec. 2015]**

Q.4 Find $f'(x)$ and $f''(x)$ at x=6 given that

x:	4.5	5	5.5	6	6.5	7	7.5
$f(x)$:	9.69	12.9	16.71	21.18	26.37	32.34	39.15

[RGPV. Dec. 2014]

Q.5 For the given table find f'(x) at x = 1.0

x:	1.0	1.1	1.2	1.3
$f(x)$:	0.841	0.891	0.932	0.963

ANS: 0.5417 [[RGPV. JUNE 2007]

Q.6 Find the first and second derivatives of f(x)at x =1.2 from the Following table

x:	1	2	3	4	5
$f(x)$:	0	1	5	6	8

ANS: 15.167 [JUNE 2003]

Q.7 Use Newton's Divided difference formula , find f'(10) from the following data

x	3	5	11	27	34
f(x)	-13	23	899	17315	35606

Ans: 233

[RGPV. Dec. 2010]

Q.8 Use Newton's Divided difference formula , find f'(9) from the following data

x	5	7	11	13	17
f(x)	150	392	1452	2366	5202

Ans: 261 **[RGPV. June 2011]**

Q.9 A slider in a machine moves along a fixed straight rod. Its distance x cm along the Rod is given below for various value of the time t seconds. Find the **velocity** of the slider and its **Acceleration** when t =0.3 second.

0	0.1	0.2	0.3	0.4	0.5	0.6
30.13	31.62	32.87	33.64	33.95	33.81	33.24

Ans: Vel. = 5.34.cm/sec. Acc.=-45 .6 cm/sec.^{2}[Dec. 2004, 2013]

Q.10 The following table gives the normal weights of babies during the first 12 months of life. Find the weight of baby at the age of 7 months.

Age in months	0	2	5	8	10	12
Weights in lbs	7.5	10.25	15	16	18	21

Dec. 2014

Q.11 A rod is rotating in a plane . The following table gives the angle θ (radians) through which the rod has turned for various values of the time t seconds:

t	0.0	0.2	0.4	0.6	0.8	1.0	1.2
q	0.00	0.12	0.49	1.12	2.02	3.20	4.67

Ans: **3.82 rad/sec. , 6.75 rad/sec^2** **[June2010, Dec.2012]**

Q.12 A rocket is launched from ground its velocity in first 80 seconds as follows. Find the acceleration at t=80

$Time\ (t)$	0	10	20	30	40	50	60	70	80
$vel.\ (v)$	30	31.63	33.44	35.47	37.75	40.33	43.25	46.69	50.67

Ans: 0.38829 m/sec^2

NUMERICAL INTEGRATION: Area Bounded between the limits x_n and x_0 is called integration b/w the limits x_n and x_0.

(1) **Trapezoidal Rule:** $\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots\ldots + y_{n-1})\right]$

(2) **Simpson's 1/3 Rule: [4(odd)+2(even)]** *[divide the interval in multiple of 2]*

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots y_{n-1}) + 2(y_2 + y_4 + \ldots\ldots + y_{n-2})]$$

(3) **Simpson's 3/8 Rule: [3(1,2,4,5,7……left multiple of 3)+ 2(3,6,9……….multiple of 3)]** *[divide the interval in multiple of 3]*

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + \ldots + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + \ldots\ldots + y_{n-3})]$$

(4) **Weddle Rule:[1,5,1,6,1,5,1]** *[divide the interval in multiple of 6]*

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{10}[(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5. + y_6) + (y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11}. + y_{12}) + \ldots\ldots]$$

Note: (1) n -ordinate means $n = n-1$ in $h=(x-x_0)/n$ (2) n- equidistance intervals means $n=n-1$ (3) n -equal parts means $n=n$

Q.13 Evaluate $\int_0^1 \frac{dx}{1+x}$ by using Simpson's 1/3 rule , n=5 **[RGPV Dec. 2015]**

Q.14 Using Simpson's 1/3 rule $\int_{-3}^{3} x^4 dx$ by by taking seven ordinates. Compare with exact values.[Hint: n=6] **[Nov. 2018]**

Q.15 Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using (i)Simpson's 1/3 rule (ii) Weddle's Rule **[RGPV Dec. 2001, Dec. 2014 ,June 2015]**

Q.16 Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using Simpson's 1/3 rule, h=1/4 **[RGPV May 2019]**

Q.17 valuate $\int_0^6 \frac{dx}{1+x^2}$ by using Simpson's 3/8 rule, **[RGPV May 2019]**

Q.18 Evaluate $\int_0^{0.6} e^{-x^2} dx$ by Simpson's 1/3 -rule, in seven ordinates. **[RGPV. June 2009,June 2014] Ans: 0.5351**

Q.19 Integrate numerically $\int_0^{\pi/2} \sin x\, dx$ [(i) By Trapezoidal rule (ii) Simpson rule using **11 ordinates**. [**Ans:0.9979, 1.00000358]**

[Hint :11 ordinates= n=10 , $x_1 = \pi/20$,$y_1 = \sin \pi/20 = 0.1564$,find the values by calculator in radians, π=22/7]**[Dec.13]**

Q.20 Integrate numerically $\int_0^{\pi/2} \sqrt{\cos x}\, dx$ [Hint : n=10 , $x_1 = \pi/20$,$y_1 =$ 0.9938,find the values by calculator in radians, π=22/7]

[Dec.13] Ans: 1.1936

Q.21 Evaluate $\int_{0.5}^{0.7} x^{1/2} e^{-x} dx$ approximately by using a suitable formula.**[RGPV. June 03, Dec. 06] Ans: 0.08482711**

Q.22 Find an approximate value of the $\log_e 5$ by calculating to four decimal places by Simpson's 1/3 rule $\int_0^5 \frac{dx}{4x+5}$

Dividing the range into 10 equal parts. [Hint: n=10] **[RGPV. June 04, Dec. 08] Ans: 0.40252**

Q.23 Find the value of $\int_1^2 \frac{dx}{x}$ by Simpson's 3/8 rule .Hence obtain approximation value of $\log_e 2$ (i) h=05 (ii) h=0.25

[RGPV. June 2006, June 2013] Ans: 0.693125

Q.24 The following table gives the velocity *v of a* particle at time *'t'*. Find the distance moved by the particle in 12 second and also find acceleration at t=2 sec.

t(sec.)	0	2	4	6	8	10	12
V(m/sec)	4	6	16	34	60	94	136

Ans: s = 536 meters, a= 3 m/sec.^{2}
[RGPV. Feb. 2010]

Q.25 A curve is drawn to pass through the points given by the following table . estimate the area bounded by the curve , x-axis and the lines x=1 , x=4.

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Ans: 7.7833 sq. unit
Hint: x_0=1, x_n=4, *[June 2007]*

Q.26 A river is 80 meters wide. The depth d(in feet) of the river at a distance x from the bank is given by the following table . Find approximately the area of cross-section of the river using Simpson's 3/8 rule..

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	12	14	8	3

Ans: 710 sq. Feet June 2010]

Solution of Algebraic Simultaneous linear equations:

Linear Algebraic Equations: Let system of linear equations is:

$$a_1x+b_1y+c_1z=d_1, \; a_2x+b_2y+c_2z=d_2, \; a_3x+b_3y+c_3z=d_3$$

1. Direct Methods:

(i) Gauss Elimination method (Method of Pivoting) : In essence, we wish to eliminate unknowns from the equations by a sequence of algebraic steps.

Let augmented matrix **[A:b]=** $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$

Normalization (i) Let $a_1 \neq 0$. Then by $27x+6y-z=85$, $6x+15y+2z=72$, $x+y+54z=110$

and $R_{31}(\frac{-a_3}{a_1}) \Rightarrow R_3 = R_3 - \frac{a_3}{a_1} R_1$,we get $\approx \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & b_3' & c_3' & d_3' \end{array}\right]$ here a1 is called **pivoting element**.

Reduction : Now take b_2' ($\neq 0$) as the pivoting element , and use $R_{32}(\frac{-b_3'}{b_2'}) \Rightarrow R_3 = R_3 - \frac{b_3'}{b_2'} R_2$

We get $\approx \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & 0 & c_3'' & d_3'' \end{array}\right]$ after solving this matrix by back substitution we get required results.

Note: This method fails if a_1, b_2' or c_3'' becomes zero. In such cases by inter changing the rows we can get the non zero pivots.

(ii) Gauss Jordan Method: It is a variation of Gauss elimination. The differences are:
- When an unknown is eliminated from an equation, it is also eliminated from all other equations.
- All rows are **normalized by dividing them by their pivot element.**

Hence, the elimination step results in an identity matrix rather than a triangular matrix. Back substitution is, therefore, not necessary.
All the techniques developed for Gauss elimination are still valid for Gauss-Jordan elimination.

GAUSS-JORDAN ELIMINATION:

1. Get a 1 in upper left corner (by row ops 1 and/or 2)
2. Get 0's everywhere else in its column (by row op 3)
3. Mentally delete row 1 and column 1. What remains is a smaller **submatrix**.
4. Get 1 in upper left-hand corner of the *sub matrix*.
5. Get 0's everywhere else in its column for *all rows* in the matrix (not just the submatrix).
6. Mentally delete row 1 and column 1 of the submatrix, forming an even smaller submatrix.
7. Repeat 4, 5, 6 until you can go no further.
8. The matrix will now be in **reduced row-echelon form** (RREF), or just **reduced form**.
6. Re-write the system in natural form.
7. State the solution.

A. If you get a row of all zeros, use row op 1 to make it the last row
B. If you get a row with all zeros to the left of the line, and a non-zero on the right, STOP (no solution).

(ii) LU Factorization Method(or Crout's Method , or Choleskey's Method)
For a nonsingular matrix [*A*] on which one can successfully conduct the Naïve Gauss elimination forward elimination steps, one can always write it as
Step –I TAKE [A]=[L][U]

Where : [L]= Lower triangular matrix with unit diagonal $=\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}$, [U] = Upper triangular matrix$=\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$

Step –II : Take [L][Z]=[b]
Step-III . [U][X]=[Z] Where Z=[z_1, z_2, z_3]
Step-IV : Use back Substitution to find values of *x*, *y*, *z*

ITRATIVE METHODS FOR SOLVING SIMULTANEOUS LINEAR EQUATION:
(i) ***Jacobi Method*** **:** Let system of equations is

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n &= b_n
\end{aligned}$$

Solve each equation for one variable:
For first equation $|a_{11}| > |a_{11}| + |a_{12}| \dots\dots + |a_{1n}|$, For Second equation $|a_{22}| > |a_{21}| + |a_{23}| \dots\dots + |a_{2n}|$ ……

$$\begin{aligned}
x_1 &= \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n\right)\right]\\
x_2 &= \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x1 + a_{23}x_3 + \dots + a_{2n}x_n\right)\right]\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
.x_n &= \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1 + a_{n2}x_2 + \dots + a_{n,n-1}x_{n-1}\right)\right]
\end{aligned}$$

The Iteration formulas are

$$\boxed{\begin{aligned}
x_1^{(i+1)} &= \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2^{(i)} + a_{13}x_3^{(i)} + \dots + a_{1n}x_n^{(i)}\right)\right]\\
x_2^{(i+1)} &= \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1^{(i)} + a_{23}x_3^{(i)} + \dots + a_{2n}x_n^{(i)}\right)\right]\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
x_n^{(i+1)} &= \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1^{(i)} + a_{n2}x_2^{(i)} + \dots + a_{n,n-1}x_{n-1}^{(i)}\right)\right]
\end{aligned}}$$

Gauss-Seidel Method:
In most cases using the newest values on the right-hand side equations will provide better estimates of the next value. If this is done, then we are using the Gauss-Seidel Method:
The Iteration formulas are:

$$x_1^{(i+1)} = \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2^{(i)} + a_{13}x_3^{(i)} + \dots + a_{1n}x_n^{(i)}\right)\right]$$
$$x_2^{(i+1)} = \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1^{(i+1)} + a_{23}x_3^{(i)} + \dots + a_{2n}x_n^{(i)}\right)\right]$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$x_n^{(i+1)} = \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1^{(i+1)} + a_{n2}x_2^{(i+1)} + \dots + a_{n,n-1}x_{n-1}^{(i)}\right)\right]$$

Note: 1. ***Why use Jacobi ? Ans:*** Because you can separate the n-equations into n independent tasks; it is very well suited to computers with parallel processors.

2. *If either method converges, Gauss-Seidel converges faster than Jacobi.*

Solution of Simultaneous Linear Algebraic Equations

Q.27 Solve by **Gauss elimination** method $10x+y+2z=13,\ 3x+10y+z=14,\ 2x+3y+10z=15$

Q.28 Solve by **Gauss elimination** method $6x+3y+2z=6,\ 6x+4y+3z=0, 20x+15y+12z=0$ **[Nov. 18]**

Q.29 Solve by **Gauss elimination** method $x+y+z=9$, $2x-3y+4z=13$, $3x+4y+5z=40$

Q.30 Apply **factorization method** to solve the equation 10x +y + 2z = 13 , 3x +10y + z = 14, 2x +3y + 10z = 15

Q.31 Apply **factorization method** to solve the equation :3x +2y + 7z = 4 , 2x +3y + z = 5, 3x +4y + z = 7[**Dec.04, 07,12**] **Ans:7/8,9/8,-1/8**

Q.32 Solve by **Gauss-Seidel method** the equations : 20x +y - 2z = 17 , 3x +20y - z = -18, 2x -3y + 20z = 25[**Ans: 1.0004,-1.00025,0.9999]**

Q.33 Solve by **Gauss-Seidel method** the equations 10x +2y +z = 9 , 2x +20y -2 z = -44, 2x -3y + 20z = 25

Q.34 Solve by **Gauss-Seidel method** the equations 20x +y - 2z = 17 , 3x +20y - z = -18, -2x +3y +10z =22 [Dec.2003]**[Ans: 1,-2,3]**

Q.35 Solve by **Gauss-Seidel method** the equations $27x+6y-z=85$,$6x+15y+2z=72,\ x+y+54z=110$

[Ans :1.05564,1.3663,1.56199] **[Dec.02,06,11 ,June 08,11,13,17Nov. 18]**

Q.36 Solve by **Jacobi's** method the equation 10x +2y +z = 9 , 2x +20y -2 z = -44, 2x -3y + 20z = 25

Q.37 Solve by **Jacobi's** method the equation 10x +y + 2z = 13 , 3x +10y + z = 14, 2x +3y + 10z = 15

Q.38 Solve by **Crouts** method $x+y+z=3$, $2x-y+3z=16$,$3x+y-z=-3$ **[May 2019]**

Q.39 Apply **crouts method** to solve the equation 10x +y + z = 12 ,2x +10y + z = 13, 2x +2y + 10z = 14 [**Ans:1,1,1**] **[Dec. 13,16,june16]**

RELAXATION METHOD

Solve the system of linear algebraic equation using relaxation Method

$$10x-2y-3z=205,\ -2x+10y-2z=154\ ,\ -2x-y+10z=120$$

Solution: The Residuals are

$$R_x = 205-10x+2y+3z \text{ ....(1)}$$
$$R_y = 154+2x-10y+2z \text{ ....(2)}$$
$$R_z = 120+2x+y-10z \text{ .... (3)}$$

Operation Table

	δRx	**δRy**	**δRz**	
δx	-10	2	2	Diff eq.(1),(2),(3) w.r.t. *"x"* Respectively
δy	2	-10	1	Diff eq.(1),(2),(3) w.r.t. *"y"* Respectively
δz	3	2	-10	Diff eq.(1),(2),(3) w.r.t. *"z"* Respectively

In this method we reduce(minimize) residuals by giving increments to the variables. The process stop when residuals become "0" or near to "0".

Residuals Table

Operations	Values	Rx	Ry	Rz
Initially we put	*x=y=z=0*	205	154	120
Since the max. residual is 205 in Rx , hence we take approximate value of $\delta x = \frac{Rx}{\lvert a_1 \rvert}$	$\delta x = \frac{205}{\lvert -10 \rvert} = 20.5 \approx 20$ Put this value in eq.(1),(2),(3) keeping *y*, *z* constant	Since the residual i.e.205 , or eq. is $R_x = 205 - 10x + 2y + 3z$ (put the value of δx=20, keeping *y,z* constant) $205 - 10(20) = 5$	Since the residual i.e.154 , or eq. is $R_y = 154 + 2x - 10y + 2z$ (put the value of δx=20, keeping *y,z* $154 + 2(20) = 194$	Since the residual i.e.120 , or eq. is $R_z = 120 + 2x + y - 10z$ (put the value of δx=20, keeping *y,z* constant) $120 + 2(20) = 160$
Since the max. residual is 194 in Ry , hence we take approximate value of $\delta y = \frac{Ry}{\lvert b_2 \rvert}$	$\delta y = \frac{194}{\lvert -10 \rvert} = 19.4 \approx 19$ Put this value in eq.(1),(2),(3) keeping *x*, *z* constant	Since new residual i.e.5 , or New eq. becomes $R_x = 5 - 10x + 2y + 3z$) (put the value of δy, keeping *x,z* constant) $5 + 2(19) = 43$	(use new residual i.e.194 , or New eq. becomes $R_y = 194 + 2x - 10y + 2z$ put the value of δy, keeping *x,z* constant) $194 - 10(19) = 4$	(use new residual i.e.160 , or New eq. becomes $R_z = 160 + 2x + y - 10z$ put the value of δy, keeping *x,z* constant) $160 + 19 = 179$
Since the max. residual is 179 in Rz , hence we take approximate value of $\delta z = \frac{Rz}{\lvert c_3 \rvert}$	$\delta z = \frac{179}{\lvert -10 \rvert} = 17.9 \approx 18$ Put this value in eq.(1),(2),(3) keeping *x*, *z* constant	(use new residual i.e.5 , or New eq. becomes $R_x = 43 - 10x + 2y + 3z$ Put the value of δz=18 , keeping *x, y* constant $43 + 3(18) = 97$	use new residual i.e.5 , or New eq. becomes $R_y = 4 + 2x - 10y + 2z$ Put the value of δz=18 , keeping *x, y* constant $4 + 2(18) = 40$	use new residual i.e.179 , or New eq. becomes $R_z = 179 + 2x + y - 10z$ Put the value of δz=18 , keeping *x, y* constant $179 - 10(18) = -1$
Since the max. residual is 97 in Rx , hence we take approximate value of $\delta x = \frac{Rx}{\lvert a_1 \rvert}$	$\delta x = \frac{97}{\lvert -10 \rvert} = 9.7 \approx 10$ Put this value in eq.(1),(2),(3) keeping *y*, *z* constant	(use new residual i.e.97 , or New eq. becomes $R_x = 97 - 10x + 2y + 3z$ Put the value of δx=10 , keeping *y,z* constant $97 - 10(10) = -3$	use new residual i.e.40 , or New eq. becomes $R_y = 40 + 2x - 10y + 2z$ Put the value of δx=10 , keeping *y,z* constant $40 + 2(10) = 60$	use new residual i.e.-1 , or New eq. becomes $R_z = -1 + 2x + y - 10z$ Put the value of δx=10 , keeping *y,z* constant $-1 + 2(10) = 19$
Since the max. residual is 60 in Ry , hence we take approximate value of $\delta y = \frac{Ry}{\lvert b_2 \rvert}$	$\delta y = \frac{60}{\lvert -10 \rvert} = 6$ Put this value in eq.(1),(2),(3) keeping *x*, *z* constant	(use new residual i.e.-3 , or New eq. becomes $R_x = -3 - 10x + 2y + 3z$ Put the value of δy=6 , keeping *x, z* constant $-3 + 2(6) = 9$	use new residual i.e.60 , or New eq. becomes $R_y = 60 + 2x - 10y + 2z$ Put the value of δy=6, keeping *x, z* constant $60 - 10(6) = 0$	use new residual i.e. 19 , or New eq. becomes $R_z = 19 + 2x + y - 10z$ Put the value of δy=6, keeping *x, z* constant $19 + 6 = 25$
Since the max. residual is 25 in Rz , hence we take approximate value of $\delta z = \frac{Rz}{\lvert c_3 \rvert}$	$\delta z = \frac{25}{\lvert -10 \rvert} = 2.5 \approx 2$ Put this value in eq.(1),(2),(3) keeping *x*, *y* constant	(use new residual i.e9 , or New eq. becomes $R_x = 9 - 10x + 2y + 3z$ Put the value of δz=2 , keeping *x, y* constant $9 + 3(2) = 15$	use new residual i.e.0 , or New eq. becomes $R_y = 0 + 2x - 10y + 2z$ Put the value of δz=2 , keeping *x, y* constant $0 + 2(2) = 4$	use new residual i.e. 25 , or New eq. becomes $R_z = 25 + 2x + y - 10z$ Put the value of δz=2 , keeping *x, y* constant $25 - 10(2) = 5$
Similarly	δx=2	-5	8	9
	δz=1	-2	10	-1
	δy=1	0	0	0

Since all the residuals are zero(or may be near equal to zero) hence we stop the process , also x

$$x = \sum \delta x = 20 + 10 + 2 = 32 \; , y = \sum \delta y = 19 + 6 + 1 = 26 \; , z = \sum \delta z = 18 + 2 + 1 = 21$$